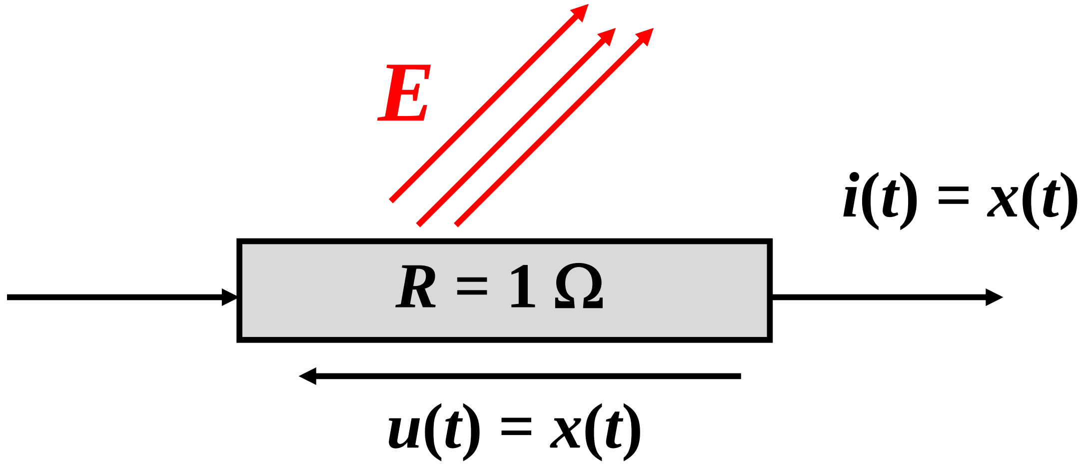
Energy/Power Properties of Signals

- Definitions of energy and power of a signal
- Signal energy and power in the frequency domain
- Autocorrelation function of an energetic signal
- Autocorrelation function of a power signal
- Taxonomy of signals
- Acf and energy/power spectrum properties
- Acf definitions – various signals
- Acf as a measure of signal similarity

Signal Energy Definition



$$E=\int_{-\infty}^{+\infty}u(t)i(t)dt=\int_{-\infty}^{+\infty}x^2(t)dt,\quad x(t)\in\mathcal{R}$$

Signal is energetic whenever *E* < ∞.

Energetic signals have spectral representation (Fourier transform does exist).

$$X(\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

Signal energy in the frequency domain

Parseval theorem

$$x(t) \leftrightarrow X(\omega),\ x(t) \in \mathcal{R}$$

$$E = \int_{-\infty}^{+\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} |X(\omega)|^2\,d\omega$$

According to the Parseval theorem signal energy can be determined either in the time or frequency domain.

Spectral energy density (energy spectrum):

$$S(\omega) = |X(\omega)|^2$$

indicates how the total signal energy is distributed over frequencies.

Parseval theorem - example

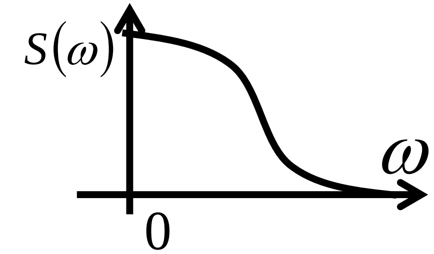
According to the Parseval theorem signal energy can be determined either in the time or frequency domain.

Find the energy of the signal *x*(*t*) in the time and frequency domain. $x(t) = \exp(-|t|)$

Signal energy in a frequency bandwidth

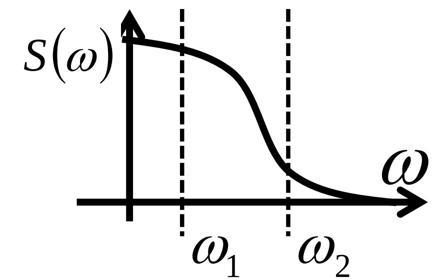
Total energy

$$E = E(0,\infty) = \frac{1}{\pi}\int_{0}^{+\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_{0}^{+\infty} S(\omega) d\omega$$



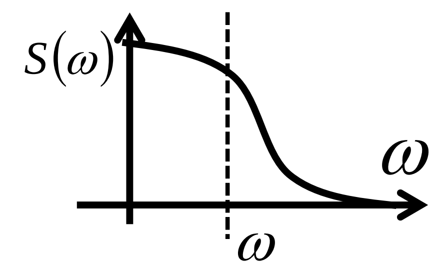
Energy in a bandwidth

$$E(\omega_1,\omega_2) = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} S(\omega) d\omega$$



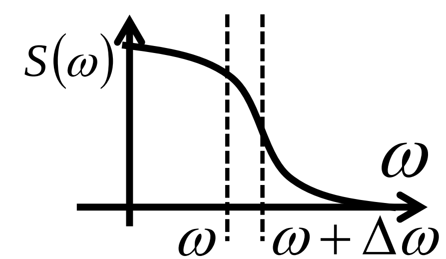
Energy up to some frequency

$$E(\omega) = \frac{1}{\pi}\int_{0}^{\omega} |X(v)|^2 dv = \frac{1}{\pi}\int_{0}^{\omega} S(v) dv$$



Energy in a narrow bandwidth

$$E(\omega) = \frac{1}{\pi}\int_{\omega}^{\omega+\Delta\omega} |X(v)|^2 dv = \frac{1}{\pi} X(\omega)\Delta\omega$$



Fractional energy

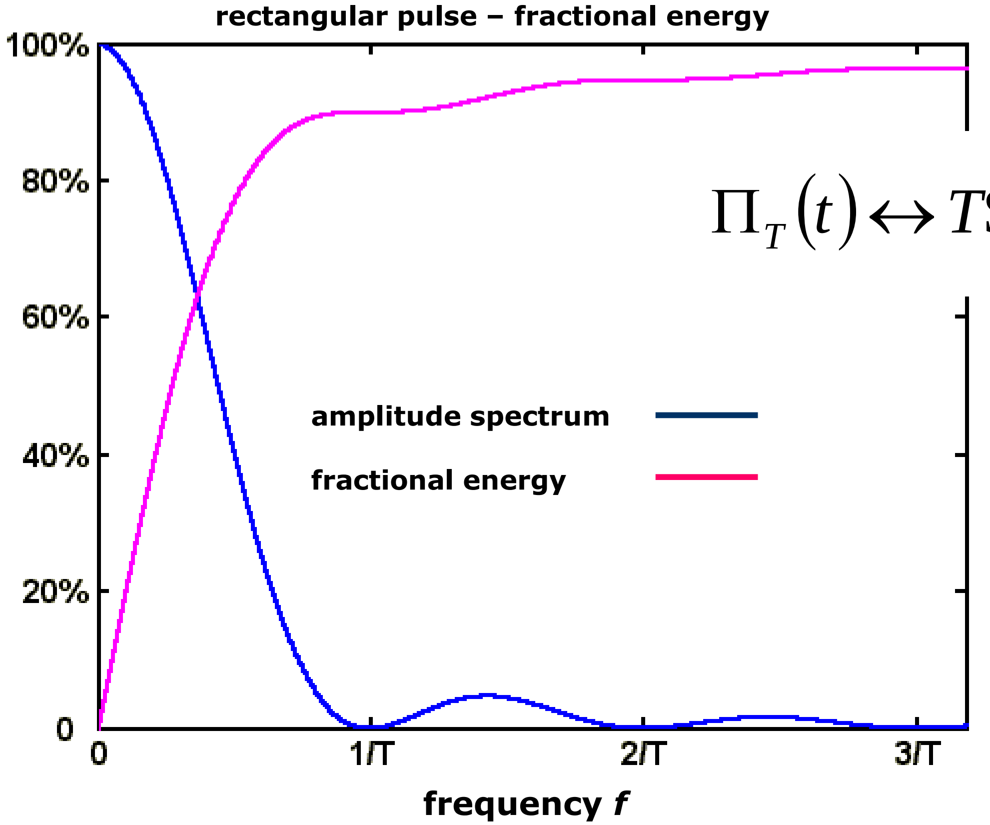
$$E(\omega)=\frac{1}{\pi}\int_0^{\omega}\left|X(v)\right|^2 dv,\ \omega\geq 0$$

$$E=\frac{1}{\pi}\int_0^{\infty}\left|X(v)\right|^2 dv$$

$$E_{\mathrm{f}}(\omega)=E(\omega)/E\xrightarrow{\omega\to\infty}1$$

Fractional energy $E_f(\omega)$ indicates the fraction of the total signal energy contained in a bandwidth [0, ω].

Fractional energy



$$\Pi_T(t) \leftrightarrow T\mathrm{Sa}\frac{\omega T}{2} = T\mathrm{Sa}\,\pi f T$$

$$E_{\mathrm{f}}(f) = \frac{2}{\pi}\int_0^{\pi f T} \mathrm{Sa}^2(v)\,dv,\ f \geq 0$$

Autocorrelation function of an energetic signal

$$S(\omega)=|X(\omega)|^2 \qquad \mathcal{F}^{-1}\{S(\omega)\}=\mathcal{F}^{-1}\left\{|X(\omega)|^2\right\}=?$$

$$\mathcal{F}^{-1}\left\{|X(\omega)|^2\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)X^*(\omega)e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\left[\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau\right]^* e^{j\omega t}d\omega=\int_{-\infty}^{\infty}x(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega(t+\tau)}d\omega\right]d\tau$$

$$=\int_{-\infty}^{\infty}x(\tau)x(t+\tau)dt$$

$$S(\omega)=|X(\omega)|^2 \leftrightarrow R(\tau)=\int_{-\infty}^{\infty}x(t)x(t+\tau)dt$$

$R(\tau)$ - autocorrelation function (acf)

Spectral energy density and autocorrelation function are related via the Fourier transform.

Autocorrelation f. - example

Find the autocorrelation function (acf) of the signal *x*(*t*):

$$x(t) = \exp(-t) \quad t \geq 0$$

$$x(t) = 0 \quad t < 0$$

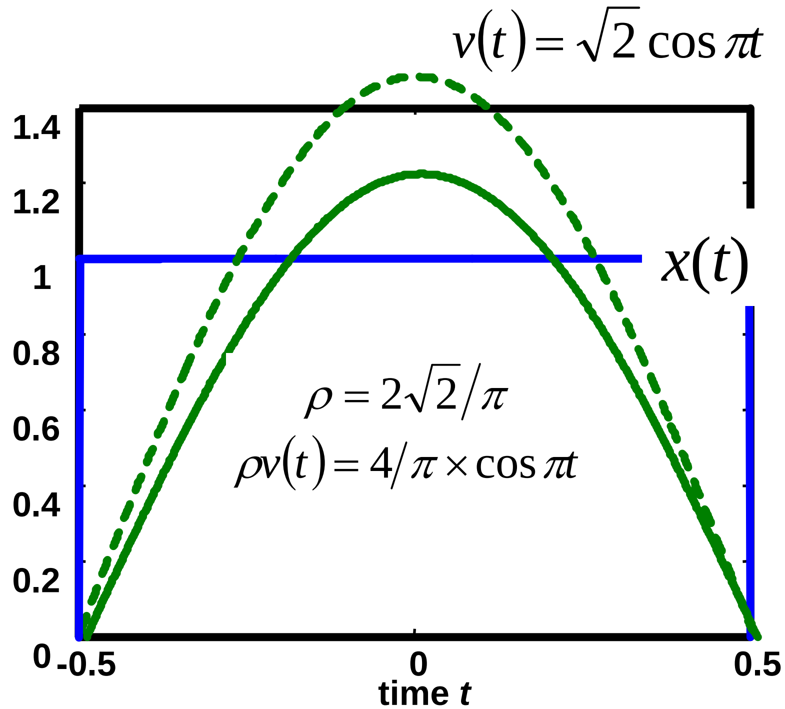
Acf – measure of signals similarity

Two signals *x*(*t*) and *v*(*t*) are given, $E_x = E_v = 1$.
Error of approximation *x*(*t*) by *ρv*(*t*) is to be minimized by proper value of coefficient *ρ*.

$$E_x = \int_{-\infty}^{+\infty} x^2(t)dt,\ E_v = \int_{-\infty}^{+\infty} v^2(t)dt = 1$$

$$e = \int_{-\infty}^{\infty} [x(t) - \rho v(t)]^2\, dt \geq 0$$

$$\min_{\rho} e \rightarrow \rho = \int_{-\infty}^{\infty} x(t)v(t)dt$$



***ρ* coefficient – interval of variation**

$$e_{\min} = 1 - \left[\int_{-\infty}^{\infty} x(t)v(t)dt\right]^2 = 1 - \rho^2 \geq 0$$

$$0 \leq |\rho| = \left|\int_{-\infty}^{\infty} x(t)v(t)dt\right| \leq 1$$

Correlation - measure of signal similarity

$$e_{\min} = 1 - \left[\int_{-\infty}^{\infty} x(t) v(t) dt \right]^2 = 1 - \rho^2 \geq 0$$

$$0 \leq |\rho| = \left| \int_{-\infty}^{\infty} x(t) v(t) dt \right| \leq 1$$

$$1 \geq e_{\min} \geq 0$$

$$\rho = 0 \Leftrightarrow e_{\min} = 1$$

$$\rho = \int_{-\infty}^{\infty} x(t) v(t) dt = 0$$

$$x(t) \perp v(t)$$

orthogonal signals

(dissimilar signals)

$$\rho = 1 \Leftrightarrow e_{\min} = 0$$

$$\rho = \int_{-\infty}^{+\infty} x(t) v(t) dt = \int_{-\infty}^{+\infty} x(t)^2 dt = 1$$

$$x(t) = \pm v(t)$$

similar signals

Correlation coefficient $\rho = \int_{-\infty}^{\infty} x(t) v(t) dt$

is a measure of similarity of two signals *x*(*t*) and *v*(*t*).

Correlation - measure of signal similarity

Normalization of *x*(*t*) and ν(*t*) necessary if $E_x \neq 1$ or $E_\nu \neq 1$.

$$\rho = \frac{\int_{-\infty}^{\infty} x(t)\nu(t)dt}{\sqrt{\int_{-\infty}^{\infty} x^2(t)dt}\sqrt{\int_{-\infty}^{\infty} \nu^2(t)dt}}$$

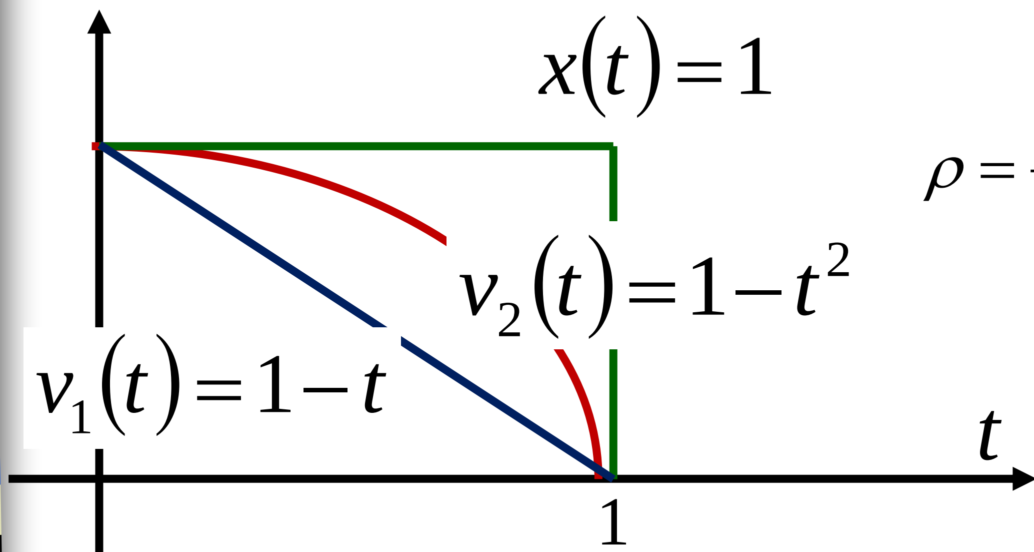
Schwarz integral inequality:

$$\left[\int_{-\infty}^{\infty} x(t)\nu(t)dt\right]^2 \leq \int_{-\infty}^{\infty} x^2(t)dt \int_{-\infty}^{\infty} \nu^2(t)dt$$

results in: $|\rho| \leq 1$

The ρ coefficient is called a correlation (similarity) coefficient for two signals *x*(*t*) and ν(*t*).

Correlation coefficient - example



$$\rho = \frac{\int_{-\infty}^{\infty} x(t)v(t)dt}{\sqrt{\int_{-\infty}^{\infty} x^2(t)dt}\sqrt{\int_{-\infty}^{\infty} v^2(t)dt}}$$

$$\rho_1 \approx 0.71$$

$$\rho_2 \approx 0.91$$

$$v(t) = \cos\frac{\pi}{2}t \quad -1 \le t \le +1$$

$$x(t) = \Lambda_2(t) \sim \cos\frac{\pi}{2}t$$

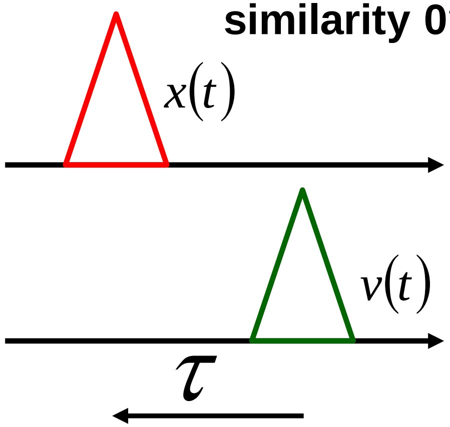
$$\rho = ?$$

WORK IN PROGRESS

Correlation - measure of signal similarity

Correlation coefficient has to involve a time delay (shift):

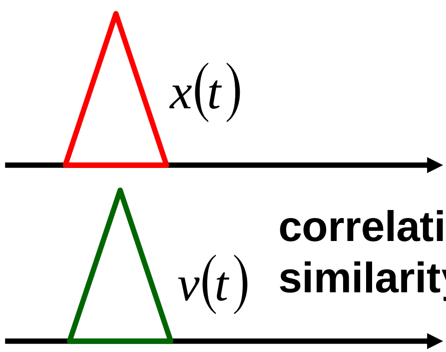
correlation = 0
similarity 0? or 1?



$$\rho(\tau)=\frac{\int_{-\infty}^{\infty} x(t)v(t+\tau)dt}{\sqrt{\int_{-\infty}^{\infty} x^2(t)dt}\sqrt{\int_{-\infty}^{\infty} v^2(t)dt}}$$

Intercorrelation function of two signals $x(t)$ and $v(t)$:

$$R_{xv}(\tau)=\int_{-\infty}^{\infty} x(t)v(t+\tau)dt$$

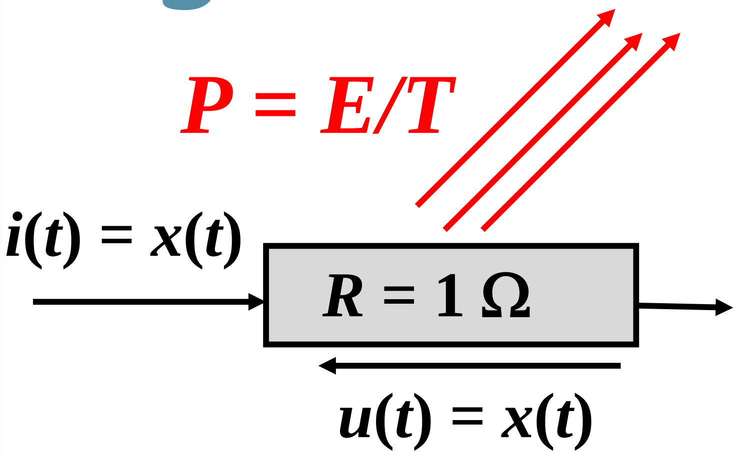


correlation = 1
similarity = 1

Autocorrelation (selfsimilarity) function of signal $x(t)$:

$$R_x(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

Signal Power Definition



$$P_T = E/T = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt, \quad x(t) \in \mathcal{R}$$

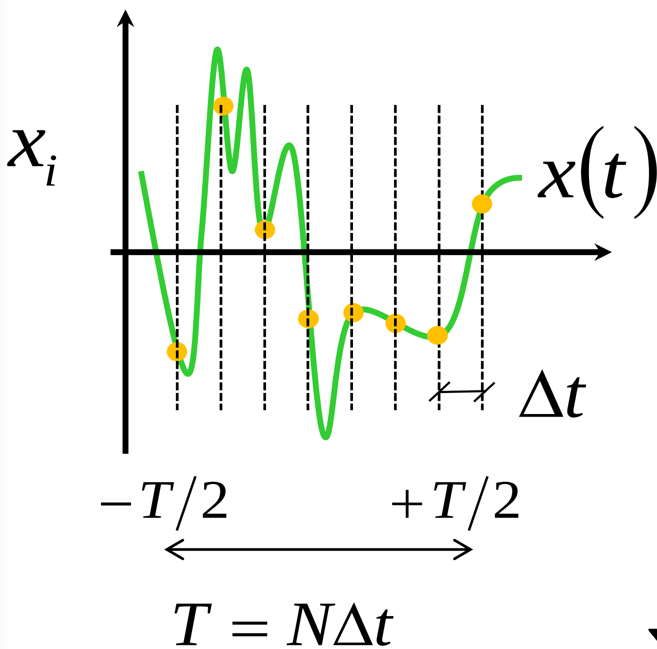
$$\underbrace{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} (\cdot)\,dt}_{\text{time averaging of } (\cdot)} = \langle \cdot \rangle$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt = \langle x^2 \rangle$$

Power signals meet the condition $P < \infty$.
Power signals do not possess a Fourier transform (unless concept of the Dirac delta is adopted).
Most popular <u>deterministic power signals are periodic signals</u>.
However, some other power signals: sgn(*t*), 1(*t*), *dc* … exist.

Time averaging (what does it mean?)

$$\langle x \rangle = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,dt$$



x_i – signal sample
i – sample number
N – # all samples
Δt – sampling interval

Sampling has nothing in common with the sampling theorem. The shorter Δt, the better time averaging.

$$\frac{\sum_i x_i}{N} = \frac{1}{T}\sum_i x_i \Delta t = \underbrace{\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,dt}_{\text{time average } x} = \langle x \rangle$$

Time averaging may be applied to any signal or its function. Example: time averaging $\langle x^2 \rangle$ provides a power of a signal while $\langle x \rangle$ yields a signal average.

Power of a periodic signal

$$x(t) = x(t + T_0), \quad T_0 \text{ - period of a signal}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)dt = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t)dt$$

Power of a periodic signal is equal to its power calculated over its single period.

Time averaging of any periodic quantity (over an infinite horizon) can be always calculated as a time average over a single period.

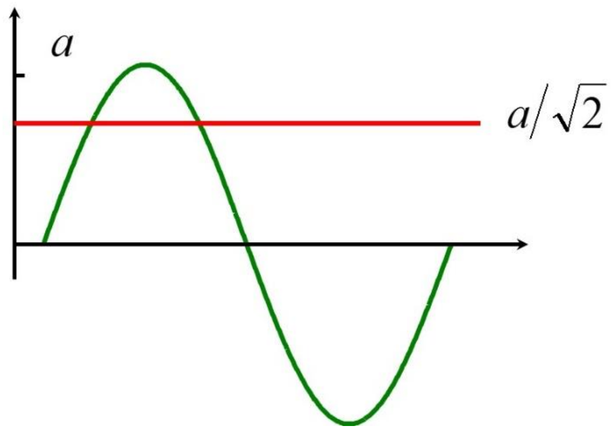
$$q(t) = q(t + T_0), \quad T_0 \text{ - period of a quantity}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} q(t)dt = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} q(t)dt$$

$$T \to \infty \equiv T := nT_0 \to \infty \equiv n \to \infty$$

Heine's definition of a limit

Power of an harmonic signal



$$x(t) = a\sin(\omega t + \varphi)$$

$$P = \frac{1}{2}a^2 = \left(a/\sqrt{2}\right)^2$$

$a/\sqrt{2}$ - **root mean square value (rms)**

rms – equivalent of a constant signal in terms of a power (energy).

Power of a harmonic depends neither on its frequency nor initial phase.

Power of a dc signal

$$x(t) = A$$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} A^2 dt = A^2$$

Signal power in the frequency domain

Parseval theorem - analogue

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)dt = \frac{1}{\pi} \int_{0}^{+\infty} S(\omega)d\omega$$

Power spectral density

$$S(\omega) \leftrightarrow R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau)dt$$

$$R(\tau) = \langle x(t)x(t+\tau) \rangle$$

$R(\tau)$ - autocorrelation function

Power spectral density – like the energy spectral density – is determined by an autocorrelation function. Note the difference in the definitions of the two acf functions.

Autcorrelation function of a periodic power signal

The only known <u>deterministic power signals are periodic signals</u>. The autocorrelation function:

$$S(\omega) \leftrightarrow R(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau)dt$$

Periodic signals have Fourier series:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jn\omega_0 t) \quad R(\tau) = ?$$

$$x(t) = a_1 \cos\omega_0 t, R(\tau) = ?$$

$$x(t) = a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t, R(\tau) = ?$$

Power of a periodic signal in frequency domain

Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

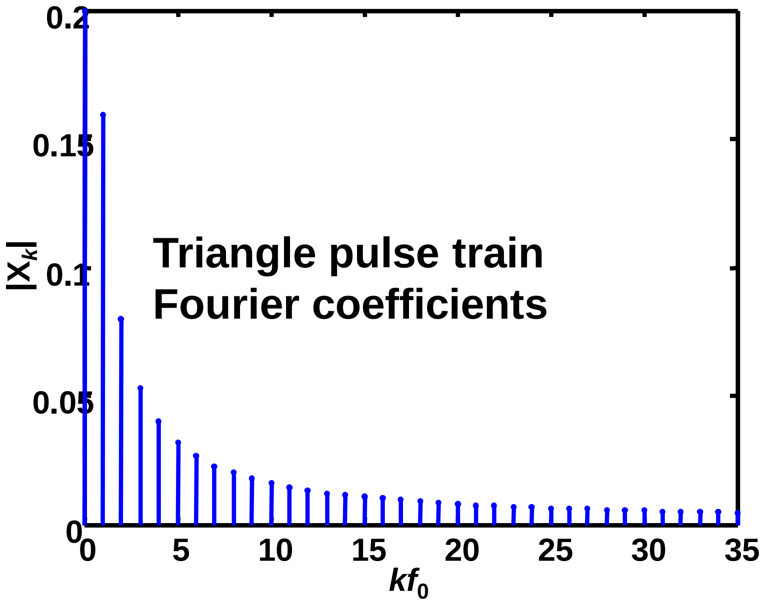
Parseval Theorem

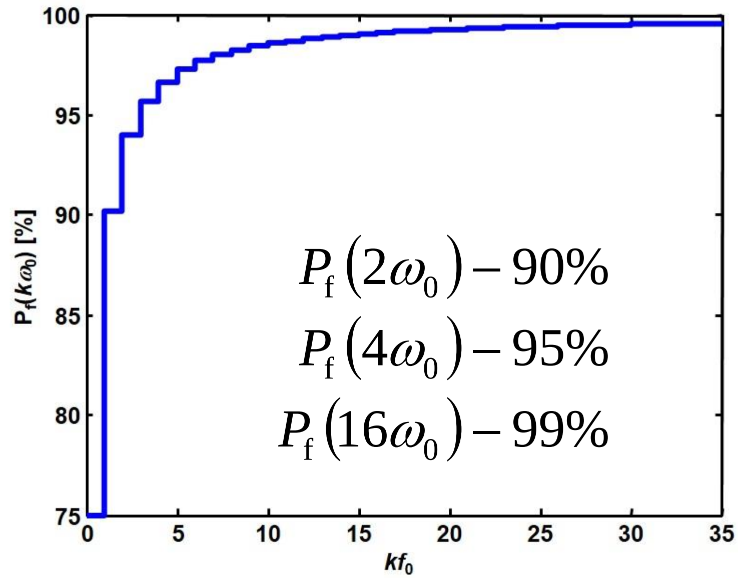
$$P = \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t)\,dt = \sum_{n=-\infty}^{+\infty} \left|X_n\right|^2$$

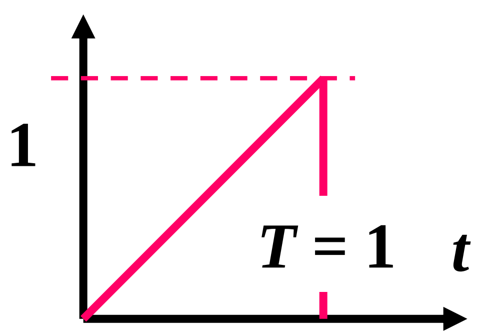
Fractional power

$$P_{\mathrm{f}}(K\omega_0) = \sum_{n=-K}^{+K} \left|X_n\right|^2 \Bigg/ \underbrace{\sum_{n=-\infty}^{+\infty} \left|X_n\right|^2}_{\text{Total power of a signal}} \xrightarrow{K\to\infty} 1$$

Fractional Power - triangle pulse train

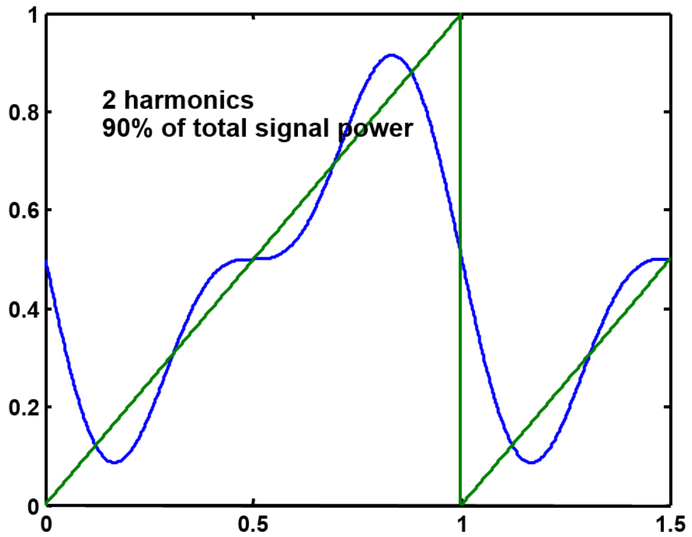


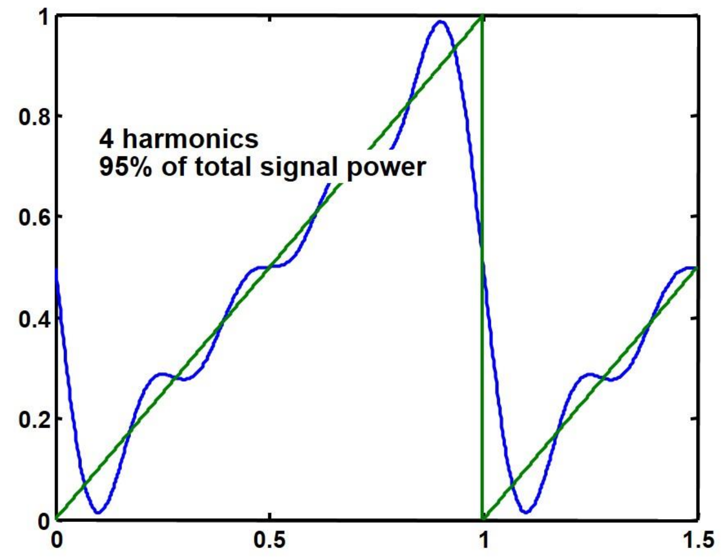


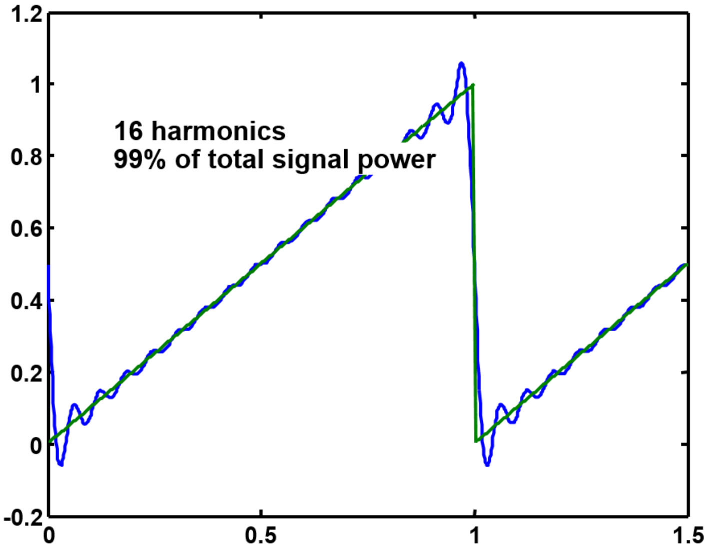


$$x(t) = \frac{1}{2} + \frac{j}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n} e^{jn2\pi t} = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n2\pi t)$$

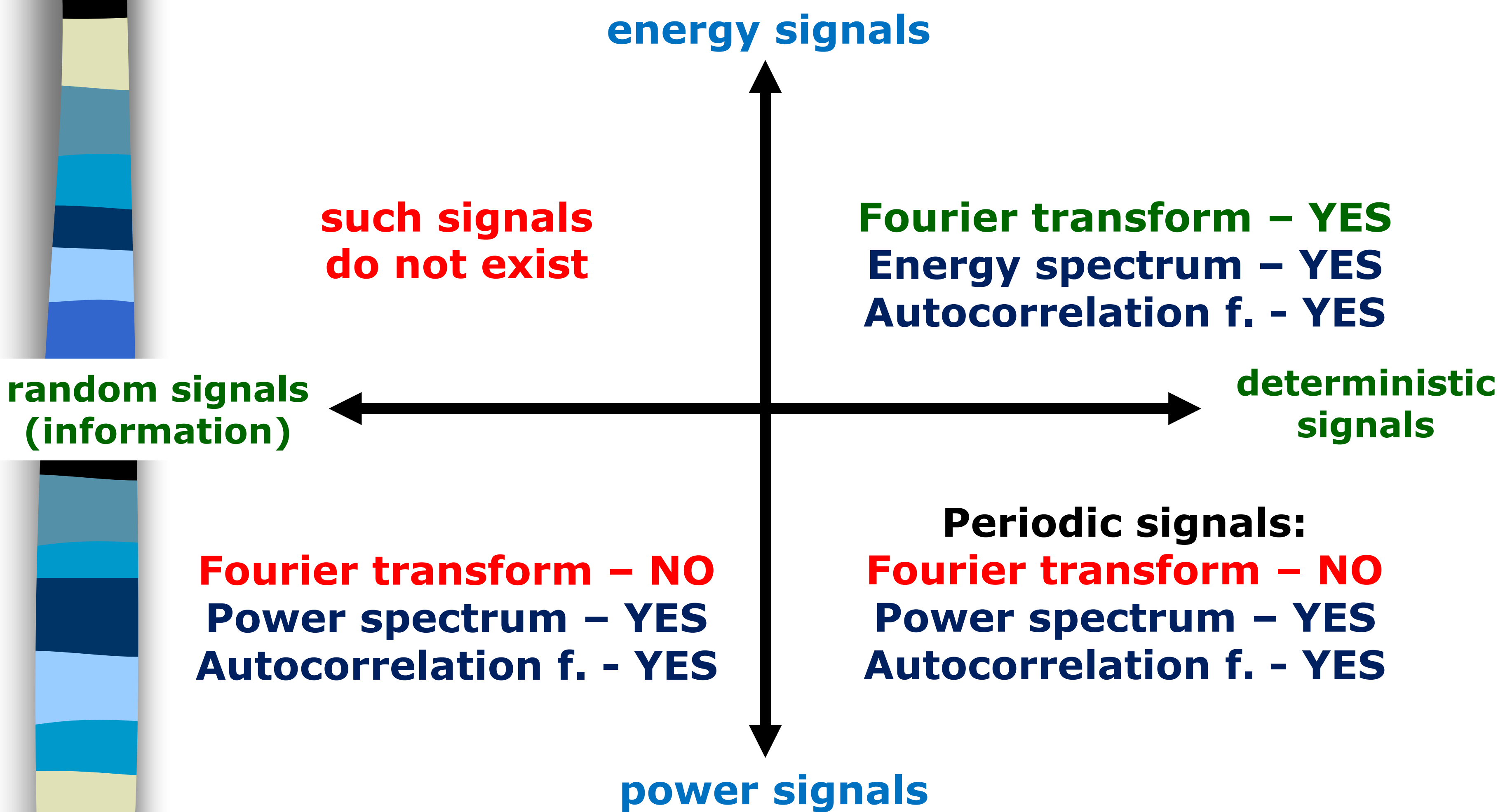
Fractional Power - triangle pulse train







Taxonomy of signals



Acf and spectrum properties

Concept of acf and power (energy) spectrum embraces spectral analysis of all signals types.

Energy (power) spectrum describe how the entire energy (power) is distributed over frequencies (bandwidth).

1. Energy (power) spectrum is a Fourier transform of an autocorrelation function (acf):

$$R(\tau) \leftrightarrow S(\omega)$$

2. Energy (power) spectrum and acf are even functions:

$$S(\omega) = S(-\omega) \qquad R(\tau) = R(-\tau)$$

Acf and spectrum properties

3. Autocorrelation function $R(\tau = 0)$ is equal to either energy or power of a signal:

$$R(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad R(0)=E$$

deterministic energy signal

$$R(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)dt \qquad R(0)=P$$

deterministic power signal

Autocorrelation value is bounded by either energy or power

$$|R(\tau)| \le R(0) = E$$

$$|R(\tau)| \le R(0) = P$$

Hint: consider the expression:

$$\int_{-\infty}^{+\infty}\left[x(t+\tau)\pm x(t)\right]^2 dt \ge 0$$

so acf may be a measure of signal selfsimilarity (similarity between a signal and its time-delayed replica).

Acf and input/output spectrum properties

$x(t)$ $\qquad$ $y(t)$

$R_x(\tau) \leftrightarrow S_x(\omega)$ $\qquad$ $R_y(\tau) \leftrightarrow S_y(\omega)$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$|Y(j\omega)|^2 = |H(j\omega)X(j\omega)|^2$$

$$S_y(\omega) = |H(j\omega)|^2 S_x(\omega)$$

4. When filtering the energy (power) spectrum is modified by a squared amplitude-frequency filter characteristic. Phase-frequency characteristic does not influence the energy (power) spectrum.

Acf and spectrum properties

The signal *x*(*t*) is input to the ideal lowpass filter with the bandwidth ω_b. Find the bandwidth ω_b such that the output signal energy is equal to ½ of the input signal energy.

$$x(t) = \beta / \left(t^2 + \alpha^2\right)$$

The autocorrelation function of some signal is given by:

$$R_x(\tau) = \exp\left(-\alpha|\tau|\right), \quad \alpha > 0$$

Find the power spectrum of the process and the fractional power of the process.
Find the bandwidth of the proces that contains 50% of the total process power.

Acf definitions – various signals

Autocorrelation function (acf)

Energy signals

$$R(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)dt$$

Acf + time averaging

Power deterministic signals

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau)dt = \langle x(t)x(t+\tau)\rangle$$

Acf + time and ensamble averaging

Power random signals

$$R(\tau) = \mathrm{E}\{\langle x(t)x(t+\tau)\rangle\}$$

Summary

1. Energy or power of a signal can be determined from energy or power spectrum, respectively.
2. Energy and power spectra are Fourier transforms of corresponding autocorrelation functions acf.
3. Autocorrelation function is defined in a more or less same manner in different signal classes.
4. Energy and power spectra posses identical properties being independent of a signal class.
5. Filtration modifies both energy and power spectra by squared amplitude-frequency function (a-f).
6. Autocrrelation function is a mesure of signal similarity.